

Number Theory I Homework Questions

1. Let n be a positive integer. Show that $\varphi(n) = \frac{n}{2}$ iff $n = 2^k$ for some positive integer k .
2. Let n, d be positive integers. Show that if $d \mid n$ then $\varphi(d)$ divides $\varphi(n)$.
3. Let n, d be positive integers. Show that if every prime divisor of d divides n then $\varphi(dn) = d\varphi(n)$.
4. Show that $\varphi(n^2) = n\varphi(n)$ for all positive integers n .
5. Find all positive integers n such that $\varphi(n) = 2$.
6. Show that there exist no integer n such that $\varphi(n) = 3$.
7. Let n, m be positive integers and let p be prime. Show that if $\gcd(n, m) = p$ then

$$\varphi(n)\varphi(m) = (p-1)\varphi\left(\frac{nm}{p}\right).$$

8. Let n, m be positive integers and let p be prime. Show that if $\gcd(n, m) = p^2$ then

$$\varphi(n)\varphi(m) = p(p-1)\varphi\left(\frac{nm}{p^2}\right).$$

9. Find all integers x such that $100 \leq x \leq 500$ and $4 \mid x, 3 \mid x+1, 5 \mid x+3$.
10. Find all integers x such that $100 \leq x \leq 300, 43!x \equiv 1 \pmod{51}$.
11. Find the smallest positive integer x such that $65!x \equiv 5 \pmod{71}$.
12. Find the largest negative integer x such that $65!x \equiv 5 \pmod{71}$.
13. Find the all odd integers x such that $100 \leq x \leq 500$ and $7 \mid x, 11 \mid x+3$.
14. Let n be an integer. Show that if $\gcd(n, 105) = 1$ then 2205 divides $n^{85} - n$.
15. Let n be an integer. Show that if $\gcd(n, 22275) = 1$ then 165 divides $n^{21} - n$.

Reference: David M. Burton, Elementary Number Theory.