Number Theory I Homework Questions

1. Let a,b be non-zero integers. Show that $\gcd(b,a)=\gcd(-b,a)=\gcd(b,-a)=\gcd(a,b+ax)$, $lcm(b,a)=lcm(-b,a)=lcm(b,-a) \qquad (\forall x\in\mathbb{Z})$

2. Let $a,b,d\in\mathbb{Z}$ and suppose that $d\mid a$, $d\mid b$ and $0<d$. Show that $\gcd\left(\frac{a}{d},\frac{b}{d}\right)=\frac{1}{d}\gcd(a,b)$

3. Show that

$$a\mid b \Leftrightarrow \quad \gcd(a,b)=|a|$$

for all non-zero integers a,b .

4. Show that $4\mid n(n+1)(n+2)(n+3)$ for all integers n

5. Show that $12\mid n(n+1)(n+2)(n+3)$ for all integers n

6. Show that $6\mid n(n-5)(n+14)$ for all integers n

7. Let $a,b,c\in\mathbb{Z},\ a,b\neq 0$ .Show that If $a\mid c$ and $b\mid c$ then $lcm(a,b)$ divides c .

8. Show that the integers q and r in Euclid's division are unique . In other words, if $aq+r=b=aq'+r'$, $0\le r,r'<a$ then $r=r',\ q=q'$

9. Let $a,b\in\mathbb{Z}$ and suppose that $\gcd(a,b)=1$. Then show that $\pi(a-b)\cap\pi(a+b)=\{2\}$ or $\pi(a-b)\cap\pi(a+b)=\varnothing$.

10. How many elements does the set $\{\gcd(a,a+12):\ a\in\mathbb{Z}\}$ have? List all elements of this set.

11. Decide whether or not the following is true:

" Let a,b be non-zero integers. Then $\pi(\gcd(a,b))=\pi(a)\cap\pi(b)$."

13. Decide whether or not the following is true:

" Let a,b and d be non-zero integers.If $\pi(d)=\pi(a)\cap\pi(b)$ then $d=\gcd(a,b)$."

14. Decide whether or not the following is true:

" Let a,b and d be integers. If $d\mid a-b$ then $d\mid a^k-b^k$ for all positive integers k ".

15. Decide whether or not the following is true:

" Let a,b and d be integers. If $d\mid a^k-b^k$ for some positive integer k, then $d\mid a-b$ ".

16. Show that if 2^n-1 is prime then n is also prime.

17. Show that every odd integer is either of the form $4k+1$ or $4k+3$

18. Show that the square of any integer is either of the form $3k$ or $3k+1$.

19. Show that $3 \nmid a^2 - 2$ for all integers a.

20. Show that the cube of any integer is either of the form $9k$, $9k+1$ or $9k+8$.

21. Show that $7 \mid 2^{3n} - 1$ for any positive integer n

22. Show that $8 \mid 3^{2n} + 1$ for any positive integer n.

23. Show that the only prime of the form $n^3 - 1$ is 7.

24. Show that if $\gcd(a,b) = 1$ and $c \mid a$ then $\gcd(c,b) = 1$.

I also recommend all students taking this course to solve the chapter end problems in the book “Elemantary Number Theory” by David M. Burton.