Aşağıdaki sorulardan section 5.6'dan 16 ve 18, section 6.1'den 20, 24 ve 27, section 6.2'den 10, 22 ve 38 sorularını çözerek diğer problemleri öğrencilere homework olarak verebilir misin. Çok teşekkürler, kolay gelsin.

Section 5.6, Pages 294-295:
6, 10, 14, 16, 18, 19, 20.

Section 6.1, Pages 315-316:
Problems: 2, 10, 18, 20, 24, 27, 28, 29.

Section 6.2, Pages 324-327:
Problems: 6, 10, 14, 22, 28(a), 34, 38.

5.6 page 294

In each of Problems 13 through 17:

(a) Show that $x = 0$ is a regular singular point of the given differential equation.

(b) Find the exponents at the singular point $x = 0$.

(c) Find the first three nonzero terms in each of two solutions (not multiples of each other) about $x = 0$.

16. $xy'' + y = 0$

18. (a) Show that

$$(\ln x)y'' + \tfrac{1}{2}y' + y = 0$$

has a regular singular point at $x = 1$.

(b) Determine the roots of the indicial equation at $x = 1$.

(c) Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n(x-1)^{r+n}$ corresponding to the larger root. Take $x - 1 > 0$.

(d) What would you expect the radius of convergence of the series to be?

6.1 Page 315

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

20. $f(t) = t^2 \sinh at$

In each of Problems 21 through 24, find the Laplace transform of the given function.

24. $$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}$$

In each of Problems 25 through 28, determine whether the given integral converges or diverges.

27. $\int_1^{\infty} t^{-2} e^t \, dt$

6.2 page 324

In each of Problems 1 through 10, find the inverse Laplace transform of the given function.

10. $F(s) = \dfrac{2s - 3}{s^2 + 2s + 10}$

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

22. $y'' - 2y' + 2y = e^{-t};$

38. Suppose that

$$g(t) = \int_0^t f(\tau) \, d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$$G(s) = F(s)/s.$$

Solutions

5.6 page 294 16

(a) From the equation we see that $P(x) = x$, $Q(x) = 0$ and $R(x) = 1$. The only zero of $P(x)$ is $x_0 = 0$, so x_0 is the only singular point of the given equation.
To check if it is also regular, we have to see if the following limits are finite:

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{0}{x} = \lim_{x\to 0} 0 = 0(= p_0),$$
$$\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{1}{x} = \lim_{x\to 0} x = 0(= q_0).$$

Since both of them are finite, $\boldsymbol{x_0 = 0}$ **is a regular singular point** of the initial equation.

(b) Because $p_0 = 0$ and $q_0 = 0$, the corresponding indical equation is

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) = 0,$$

with roots

$$\boldsymbol{r_1 = 1,\ r_2 = 0}.$$

(c) By **Theorem 5.6.1**, the first solution is given by

$$y_1(x) = x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(1)x^n\right] = x\left[1 + \sum_{n=1}^{\infty} a_n(1)x^n\right]. \tag{1}$$

Its coefficients, apart from $a_0 = 1$, are determined by the recurrence relation (8) in the book, which is, in case of $r = r_1$,

$$n(n+1)a_n(1) + \sum_{k=0}^{n-1} a_k(1)[(k+1)p_{n-k} + q_{n-k}] = 0, \forall n \in \mathbb{N}, \tag{2}$$

where

$$\sum_{n=0}^{\infty} p_n x^n = xp(x) = 0 \tag{3}$$

and

$$\sum_{n=0}^{\infty} q_n x^n = x^2 q(x) = x. \tag{4}$$

(3) and (4) imply

$$p_n = 0,\ \forall n \in \mathbb{N},$$
$$q_1 = 1,\ q_n = 0,\ \forall n \in \mathbb{N}_0 \setminus \{1\}.$$

Factoring out $a_n(0)$ from (2) gives

$$a_n(1) = \frac{-\sum_{k=0}^{n-1} a_k(1)[(k+1)p_{n-k} + q_{n-k}]}{n(n+1)}, \ \forall n \in \mathbb{N},$$

with first three elements

$$\begin{aligned}
a_1(1) &= \frac{-a_0(p_1+q_1)}{1 \cdot 2} = -\frac{1}{2}, \\
a_2(1) &= \frac{-[a_0(p_2+q_2) + a_1(1)(2p_1+q_1)]}{2 \cdot 3} = \frac{-[0 - \frac{1}{2} \cdot 1]}{6} = \frac{1}{12}, \\
a_3(1) &= \frac{-[a_0(p_3+q_3) + a_1(1)(2p_2+q_2) + a_2(1)(3p_1+q_1)]}{3 \cdot 4} \\
&= \frac{-[0 - \frac{1}{2} \cdot 0 + \frac{1}{12}(0+1)]}{12} = -\frac{1}{144}.
\end{aligned}$$

Therefore, the first solution is

$$\boxed{y_1(x) = x\left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \cdots\right) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots}.$$

Acording to the same theorem, since $r_1 - r_2 = 1$, the second solution is given by

$$\begin{aligned}
y_2(x) &= a y_1(x) \ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(0) x^n\right] \\
&= a y_1(x) \ln|x| + 1 + \sum_{n=1}^{\infty} c_n(0) x^n, \qquad (5)
\end{aligned}$$

where

$$a = \lim_{r \to r_2} (r - r_2) a_N(r), \ N = r_1 - r_2.$$

Since $N = 1$ and

$$\begin{aligned}
a_N(r) = a_1(r) &= \frac{-\sum_{k=0}^{N-1} a_k[(r+k)p_{N-k} + q_{N-k}]}{(r+N-1)(r+N)} \\
&= \frac{-a_0(rp_1+q_1)}{r(r+1)} = \frac{-1}{r(r+1)},
\end{aligned}$$

we have

$$a = \lim_{r \to 0} r \frac{-1}{r(r+1)} = -1.$$

To determine $c_n(0)$, we can substitute (5) into the initial differential equation. First, we differentiate (5) two times:

$$\begin{aligned}
y_2'(x) &= a y_1'(x) \ln|x| + a\frac{y_1(x)}{x} + \sum_{n=1}^{\infty} n c_n(0) x^{n-1}, \\
y_2''(x) &= a y_1''(x) \ln|x| + 2a\frac{y_1'(x)}{x} - a\frac{y_1(x)}{x^2} + \sum_{n=2}^{\infty} n(n-1) c_n(0) x^{n-2}.
\end{aligned}$$

We also differentiate (1) once:

$$\begin{aligned}
y_1'(x) &= \left\{x\left[1 + \sum_{n=1}^{\infty} a_n(1) x^n\right]\right\}' = 1 + \sum_{n=1}^{\infty} a_n(1) x^n + x \sum_{n=1}^{\infty} n a_n(1) x^{n-1} \\
&= 1 + \sum_{n=1}^{\infty} (1+n) a_n(1) x^n
\end{aligned}$$

Now we obtain

$$
\begin{aligned}
xy_2'' + y_2 &= x\left[ay_1''\ln|x| + 2a\frac{y_1'}{x} - a\frac{y_1}{x^2} + \sum_{n=2}^{\infty} n(n-1)c_n(0)x^{n-2}\right] \\
&+ \left[ay_1\ln|x| + 1 + \sum_{n=1}^{\infty} c_n(0)x^n\right] \\
&= a\ln|x|\underbrace{(xy_1'' + y_1)}_{0} + 2ay_1' - a\frac{y_1}{x} + \sum_{n=2}^{\infty} n(n-1)c_n(0)x^{n-1} \\
&+ 1 + \sum_{n=1}^{\infty} c_n(0)x^n \\
&= 2a\left[1 + \sum_{n=1}^{\infty}(1+n)a_n(1)x^n\right] - ax\left[1 + \sum_{n=1}^{\infty} a_n(1)x^n\right]\frac{1}{x} \\
&+ \sum_{n=2}^{\infty} n(n-1)c_n(0)x^{n-1} + 1 + \sum_{n=1}^{\infty} c_n(0)x^n \\
&= 1 + a + a\sum_{n=1}^{\infty}(1+2n)a_n(1)x^n + \sum_{n=1}^{\infty} n(n+1)c_{n+1}(0)x^n \\
&+ \sum_{n=1}^{\infty} c_n(0)x^n \\
&= 1 + a + \sum_{n=1}^{\infty}[a(1+2n)a_n(1) + n(n+1)c_{n+1}(0) + c_n(0)]x^n \\
&= \sum_{n=1}^{\infty}[-(1+2n)a_n(1) + n(n+1)c_{n+1}(0) + c_n(0)]x^n = 0 \qquad (6) \\
&= 0.
\end{aligned}
$$

Equating the coefficients of powers of x on both sides in (6), we obtain

$$
c_{n+1}(0) = \frac{(1+2n)a_n(1) - c_n(0)}{n(n+1)}, \quad \forall n \in \mathbb{N},\ n \geqslant 2,
$$

or

$$
c_n(0) = \frac{(2n-1)a_{n-1}(1) - c_{n-1}(0)}{n(n-1)}, \quad \forall n \in \mathbb{N},\ n \geqslant 2.
$$

Since there are no conditions on $c_1(0)$, we can choose $c_1(0) = 0$, which then implies

$$
\begin{aligned}
c_2(0) &= \frac{3a_1(1) - c_1(0)}{2} = \frac{3(-\frac{1}{2})}{2} = -\frac{3}{4}, \\
c_3(0) &= \frac{5a_2(1) - c_2(0)}{3 \cdot 2} = \frac{5 \cdot \frac{1}{12} + \frac{3}{4}}{6} = \frac{\frac{14}{12}}{6} = \frac{7}{36} \text{ and} \\
c_4(0) &= \frac{7a_3(1) - c_3(0)}{4 \cdot 3} = \frac{7(-\frac{1}{144}) - \frac{7}{36}}{12} = \frac{-\frac{35}{144}}{12} = -\frac{35}{1728}.
\end{aligned}
$$

Therefore, the second solution is

$$
\boxed{y_2(x) = -y_1(x)\ln|x| + 1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \cdots}.
$$

5.6 page 294. 18

(a) From the equation we see that $P(x) = \ln x$, $Q(x) = \frac{1}{2}$ and $R(x) = 1$. The only zero of $P(x)$ is $x_0 = 1$, so x_0 is the only singular point of the given equation.
To check if it is also regular, we have to see if the following limits are finite:

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} = \lim_{x \to 1} \frac{(x-1)\frac{1}{2}}{\ln x} \overset{\text{L'Hôpital}}{=} \lim_{x \to 1} \frac{\frac{1}{2}}{\frac{1}{x}} = \frac{1}{2}(= p_0),$$

$$\lim_{x \to x_0} (x - x_0)^2\frac{R(x)}{P(x)} = \lim_{x \to 1} \frac{(x-1)^2}{\ln x} \overset{\text{L'Hôpital}}{=} \lim_{x \to 1} \frac{2(x-1)}{\frac{1}{x}} = 0(= q_0).$$

Since both of them are finite, $\boldsymbol{x_0 = 1}$ **is a regular singular point** of the initial equation.

(b) Because $p_0 = \frac{1}{2}$ and $q_0 = 0$, the corresponding indical equation is

$$F(r) = r(r-1) + p_0 r + q_0 = r(r-1) + \frac{1}{2}r = r\left(r - \frac{1}{2}\right),$$

with roots

$$\boxed{r_1 = \frac{1}{2},\ r_2 = 0}.$$

(c) The solution corresponding to the larger root $r_1 = \frac{1}{2}$ is given by

$$\begin{aligned} y_1(x) &= (x-1)^{1/2}\left[1 + \sum_{n=1}^{\infty} a_n\left(\frac{1}{2}\right)(x-1)^n\right] \\ &= (x-1)^{1/2} + \sum_{n=1}^{\infty} a_n\left(\frac{1}{2}\right)(x-1)^{n+\frac{1}{2}}. \end{aligned} \tag{1}$$

We will calculate the coefficients $a_n\left(\frac{1}{2}\right)$ by substituting the expression for y_1 from (1) into the initial equation. First, we differentiate y_1 two times, obtaining

$$y_1'(x) = \frac{1}{2}(x-1)^{-1/2} + \sum_{n=1}^{\infty}\left(n + \frac{1}{2}\right)a_n\left(\frac{1}{2}\right)(x-1)^{n-\frac{1}{2}} \text{ and} \tag{2}$$

$$y_1''(x) = -\frac{1}{4}(x-1)^{-3/2} + \sum_{n=1}^{\infty}\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)a_n\left(\frac{1}{2}\right)(x-1)^{n-\frac{3}{2}}. \tag{3}$$

The Taylor series for $\ln x$ around the value $x_0 = 1$ is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}. \tag{4}$$

Substituting (1), (2), (3) and (4) into the initial equation, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}\\
&\cdot\left[-\frac{1}{4}(x-1)^{-3/2}+\sum_{n=1}^{\infty}\Big(n+\frac{1}{2}\Big)\Big(n-\frac{1}{2}\Big)a_n\Big(\frac{1}{2}\Big)(x-1)^{n-\frac{3}{2}}\right]\\
&+\frac{1}{2}\left[\frac{1}{2}(x-1)^{-1/2}+\sum_{n=1}^{\infty}\Big(n+\frac{1}{2}\Big)a_n\Big(\frac{1}{2}\Big)(x-1)^{n-\frac{1}{2}}\right]\\
&+(x-1)^{1/2}+\sum_{n=1}^{\infty}a_n\Big(\frac{1}{2}\Big)(x-1)^{n+\frac{1}{2}}=\\
=&-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{n-\frac{3}{2}}}{n}\\
&+\sum_{n=0}^{\infty}(-1)^{n}\frac{(x-1)^{n+1}}{n+1}\cdot\sum_{n=0}^{\infty}\Big(n+\frac{3}{2}\Big)\Big(n+\frac{1}{2}\Big)a_{n+1}\Big(\frac{1}{2}\Big)(x-1)^{n-\frac{1}{2}}\\
&+\frac{1}{4}(x-1)^{-1/2}+\frac{1}{2}\sum_{n=1}^{\infty}\Big(n+\frac{1}{2}\Big)a_n\Big(\frac{1}{2}\Big)(x-1)^{n-\frac{1}{2}}\\
&+(x-1)^{1/2}+\sum_{n=1}^{\infty}a_n\Big(\frac{1}{2}\Big)(x-1)^{n+\frac{1}{2}}\\
=&-\frac{1}{4}(x-1)^{-\frac{1}{2}}+\frac{1}{8}(x-1)^{\frac{1}{2}}-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{n+\frac{1}{2}}}{n+2}\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\Big(n-k+\frac{3}{2}\Big)\Big(n-k+\frac{1}{2}\Big)a_{n-k+1}\Big(\frac{1}{2}\Big)(x-1)^{n+\frac{1}{2}}\\
&+\frac{1}{4}(x-1)^{-1/2}+\frac{3}{4}a_1\Big(\frac{1}{2}\Big)(x-1)^{\frac{1}{2}}+\frac{1}{2}\sum_{n=1}^{\infty}\Big(n+\frac{3}{2}\Big)a_{n+1}\Big(\frac{1}{2}\Big)(x-1)^{n+\frac{1}{2}}\\
&+(x-1)^{1/2}+\sum_{n=1}^{\infty}a_n\Big(\frac{1}{2}\Big)(x-1)^{n+\frac{1}{2}}\\
=&\left[\frac{1}{8}+1+2\cdot\frac{3}{4}a_1\Big(\frac{1}{2}\Big)\right](x-1)^{\frac{1}{2}}+\sum_{n=1}^{\infty}\left[\frac{1}{4}\frac{(-1)^n}{n+2}\right.\\
&+\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\Big(n-k+\frac{3}{2}\Big)\Big(n-k+\frac{1}{2}\Big)a_{n-k+1}\Big(\frac{1}{2}\Big)\\
&\left.+\frac{1}{2}\Big(n+\frac{3}{2}\Big)a_{n+1}\Big(\frac{1}{2}\Big)+a_n\Big(\frac{1}{2}\Big)\right](x-1)^{n+\frac{1}{2}}\\
=&\,0.
\end{aligned}$$

Equating the coefficients of powers of x on both sides in the last equation, we obtain

$$\begin{cases} \frac{9}{8} + \frac{3}{2} a_1\left(\frac{1}{2}\right) = 0, \\ \frac{1}{4}\frac{(-1)^n}{n+2} + \sum_{k=0}^{n} \frac{(-1)^k}{k+1}\left(n-k+\frac{3}{2}\right)\left(n-k+\frac{1}{2}\right)a_{n-k+1}\left(\frac{1}{2}\right) \\ +\frac{1}{2}\left(n+\frac{3}{2}\right)a_{n+1}\left(\frac{1}{2}\right) + a_n\left(\frac{1}{2}\right) = 0, \ \forall n \in \mathbb{N}. \end{cases}$$

Therefore, we have

$$a_1\left(\frac{1}{2}\right) = -\frac{9}{8}\cdot\frac{2}{3} = -\frac{3}{4},$$

and for $n = 1$

$$\begin{aligned} & -\frac{1}{12} + \frac{5}{2}\cdot\frac{3}{2}a_2\left(\frac{1}{2}\right) - \frac{3}{8}a_1\left(\frac{1}{2}\right) + \frac{1}{2}\cdot\frac{5}{2}a_2\left(\frac{1}{2}\right) + a_1\left(\frac{1}{2}\right) = \\ & = -\frac{1}{12} + 5a_2\left(\frac{1}{2}\right) + \frac{5}{8}a_1\left(\frac{1}{2}\right) = -\frac{1}{12} + 5a_2\left(\frac{1}{2}\right) - \frac{15}{32} \\ & = -\frac{8+45}{96} + 5a_2\left(\frac{1}{2}\right) = 0, \end{aligned}$$

which implies

$$a_2\left(\frac{1}{2}\right) = \frac{\frac{53}{96}}{5} = \frac{53}{480}.$$

Therefore, the first solution is

$$\boxed{y_1(x) = (x-1)^{1/2}\left[1 - \frac{3}{4}(x-1) + \frac{53}{480}(x-1)^2 + \cdots\right]}.$$

(d) As expansion of $\ln x$ around 1 has radius of convergence equal to 1, y_1 is expected to have the same radius of convergence.

6.1 Page 315 Q20

The Laplace transform of $f(t)$ is given by

$$\begin{aligned}\mathcal{L}[f(t)] = F(s) &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty t^2\sinh(at)e^{-st}dt \\ &= \lim_{A\to\infty}\int_0^A t^2\sinh(at)e^{-st}dt \\ &= \begin{bmatrix} u=t & dv = t\sinh(at)e^{-st}dt \\ du=dt & v=\int t\sinh(at)e^{-st}dt\end{bmatrix} = (*)\end{aligned}$$

Now we calculate v with the help of solution of Problem 8:

$$\begin{aligned} v &= \int t\sinh(at)e^{-st}dt \\ &= \begin{bmatrix} u^1 = t & dv^1 = \sinh(at)e^{-st}dt \\ du^1 = dt & v^1 = \int \sinh(at)e^{-st}dt = \frac{1}{2}\left(\frac{e^{(a-s)t}}{a-s} + \frac{e^{-(a+s)t}}{a+s}\right)\end{bmatrix} \\ &= \frac{t}{2}\left[\frac{e^{(a-s)t}}{a-s} + \frac{e^{-(a+s)t}}{a+s}\right] - \frac{1}{2}\int\left[\frac{e^{(a-s)t}}{a-s} + \frac{e^{-(a+s)t}}{a+s}\right]dt \\ &= \frac{t}{2}\left[\frac{e^{(a-s)t}}{a-s} + \frac{e^{-(a+s)t}}{a+s}\right] - \frac{1}{2}\left[\frac{e^{(a-s)t}}{(a-s)^2} - \frac{e^{-(a+s)t}}{(a+s)^2}\right].\end{aligned}$$

Returning back to (*) and using some calculations from Problem 17, we have

$$\begin{aligned}(*) &= \lim_{A\to\infty}\left(\frac{t^2}{2}\left[\frac{e^{(a-s)t}}{a-s} + \frac{e^{-(a+s)t}}{a+s}\right]\Bigg|_0^A - \frac{t}{2}\left[\frac{e^{(a-s)t}}{(a-s)^2} - \frac{e^{-(a+s)t}}{(a+s)^2}\right]\Bigg|_0^A\right. \\ &\quad \left. - \int_0^A\left\{\frac{t}{2}\left[\frac{e^{(a-s)t}}{a-s} + \frac{e^{-(a+s)t}}{a+s}\right] - \frac{1}{2}\left[\frac{e^{(a-s)t}}{(a-s)^2} - \frac{e^{-(a+s)t}}{(a+s)^2}\right]\right\}dt\right) \\ &= \lim_{A\to\infty}\left\{\frac{A^2}{2}\left[\frac{e^{(a-s)A}}{a-s} + \frac{e^{-(a+s)A}}{a+s}\right] - \frac{A}{2}\left[\frac{e^{(a-s)A}}{(a-s)^2} - \frac{e^{-(a+s)A}}{(a+s)^2}\right]\right. \\ &\quad - \frac{1}{2(a-s)}\int_0^A te^{(a-s)t}dt - \frac{1}{2(a+s)}\int_0^A te^{-(a+s)t}dt \\ &\quad \left. + \frac{1}{2}\left[\frac{e^{(a-s)t}}{(a-s)^3} + \frac{e^{-(a+s)t}}{(a+s)^3}\right]\Bigg|_0^A\right\} \\ &= \lim_{A\to\infty}\left\{\frac{A^2}{2}\left[\frac{e^{(a-s)A}}{a-s} + \frac{e^{-(a+s)A}}{a+s}\right] - \frac{A}{2}\left[\frac{e^{(a-s)A}}{(a-s)^2} - \frac{e^{-(a+s)A}}{(a+s)^2}\right]\right. \\ &\quad - \frac{1}{2(a-s)}\left[\frac{Ae^{(a-s)A}}{a-s} - \frac{e^{(a-s)A}}{(a-s)^2} + \frac{1}{(a-s)^2}\right] \\ &\quad - \frac{1}{2(a+s)}\left[-\frac{Ae^{-(a+s)A}}{a+s} - \frac{e^{-(a+s)A}}{(a+s)^2} + \frac{1}{(a+s)^2}\right] \\ &\quad \left. + \frac{1}{2}\left[\frac{e^{(a-s)A}}{(a-s)^3} + \frac{e^{-(a+s)A}}{(a+s)^3}\right] - \frac{1}{2}\left[\frac{1}{(a-s)^3} + \frac{1}{(a+s)^3}\right]\right\}.\end{aligned}$$

The limit exists only for $\boldsymbol{a - s < 0}$ and $\boldsymbol{a + s > 0}$, i.e. for $s > |a|$, and since $e^{\pm a - sA}$ grows faster than A when A becomes larger, we have

$$F(s) = -\frac{1}{2}\frac{1}{(a-s)^3} - \frac{1}{2}\frac{1}{(a+s)^3} - \frac{1}{2}\frac{1}{(a-s)^3} - \frac{1}{2}\frac{1}{(a+s)^3}$$
$$= -\frac{1}{(a-s)^3} - \frac{1}{(a+s)^3} = -\frac{2a^3 + 6as^2}{[(a-s)(a+s)]^3} = -\frac{2a(a^2+3s^2)}{-(s^2-a^2)^3}.$$

Therefore, the solution is

$$\boxed{F(s) = \frac{2a(a^2+3s^2)}{(s^2-a^2)^3}.}$$

6.1 Page 315 Q24

The Laplace transform of the given piecewise continuous function is

$$\boxed{F(s)} = \int_0^\infty f(t)e^{-st}dt = \int_0^1 t \cdot e^{-st}dt + \int_1^2 (2-t)\cdot e^{-st}dt + \int_2^\infty 0 \cdot e^{-st}dt$$
$$= \begin{bmatrix} u = t & dv = e^{-st}dt & | & u = 2-t & dv = e^{-st}dt \\ du = dt & v = -\frac{e^{-st}}{s} & | & du = -dt & v = -\frac{e^{-st}}{s} \end{bmatrix}$$
$$= -t\frac{e^{-st}}{s}\bigg|_0^1 + \int_0^1 \frac{e^{-st}}{s}dt - (2-t)\frac{e^{-st}}{s}\bigg|_1^2 - \int_1^2 \frac{e^{-st}}{s}dt + \lim_{A\to\infty}\int_2^A 0dt$$
$$= -\frac{e^{-s}}{s} - \frac{e^{-st}}{s^2}\bigg|_0^1 + \frac{e^{-s}}{s} + \frac{e^{-st}}{s^2}\bigg|_1^2 - \lim_{A\to\infty} 0$$
$$= -\frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}$$
$$= \boxed{\frac{1 - 2e^{-s} + e^{-2s}}{s^2}}.$$

6.1 Page 315 Q27

Consider the function $f : [1, \infty) \to \mathbb{R}$ given by $f(t) = \frac{e^t}{t^2}$.

Observe that $f'(t) = \frac{t^2 e^t - 2te^t}{t^4} = \frac{e^t}{t^3}(t-2)$ for all $t \in [1, \infty)$.

Thus, $f'(t) < 0$ for all $t \in [1, 2]$, $f'(2) = 0$ and $f'(t) > 0$ for all $t \in (2, \infty)$ which gives us that f reaches its minimum value at $t = 2$ and the minimum value of f is $\frac{e^2}{4}$.

Since $1 < \frac{e^2}{4}$ we have that $1 < \frac{e^t}{t^2}$ for all $t \in [1, \infty)$ and then since $\displaystyle\int_1^\infty 1dt = \lim_{b\to\infty} \int_1^b 1dt = \lim_{b\to\infty} \left(t|_1^b\right) = \lim_{b\to\infty} (b-1) = \infty$ we have by the comparison test that $\displaystyle\int_1^\infty t^{-2}e^t dt$ diverges.

6.1 Page 324 Q10

We know that $\mathcal{L}(e^{at} \sin bt) = \dfrac{b}{(s-a)^2 + b^2}$ and $L(e^{at} \cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}$.

Using that and that Laplace transform is linear,

$$F(s) = \frac{2s-3}{s^2+2s+10} = \frac{2s+2-5}{s^2+2s+10} = 2 \cdot \frac{s+1}{(s+1)^2+3^2} - \frac{5}{3} \cdot \frac{3}{(s+1)^2+3^2} =$$

$$= 2 \cdot \mathcal{L}(e^{-t} \cos 3t) - \frac{5}{2} \cdot \mathcal{L}(e^{-t} \sin 3t) = \mathcal{L}(2e^{-t} \cos 3t - \frac{5}{3}e^{-t} \sin 3t)$$

Inverse Laplace transform of given function is

$$\mathcal{L}^{-1}(\mathcal{L}(2e^{-t} \cos 3t - \frac{5}{3}e^{-t} \sin 3t)) = 2e^{-t} \cos 3t - \frac{5}{3}e^{-t} \sin 3t.$$

6.1 Page 324 Q22

1	$y'' - 2y' + 2y = e^{-t},\ y(0) = 0,\ y'(0) = 1$	Question
2	$L\{y''\} = s^2Y(s) - s \cdot y(0) - y'(0)$ $L\{y'\} = s \cdot Y(s) - y(0)$ $L\{y\} = Y(s)$ $L\{sin(t)\} = \dfrac{1}{s^2+1}$ $L\{cos(t)\} = \dfrac{s}{s^2+1}$ $L\{e^{at}\} = \dfrac{1}{s-a}$ $L\{e^{ct}f(t)\} = Y(s-c)$	Laplace Transforms needed
3	$s^2Y(s) - s \cdot y(0) - y'(0) - 2[s \cdot Y(s) - y(0)] + 2Y(s) = \dfrac{1}{s+1}$ $\Rightarrow\ s^2Y(s) - s \cdot 0 - 1 - 2[s \cdot Y(s) - 0] + 2Y(s) = \dfrac{1}{s+1}$ $\Rightarrow\ s^2Y(s) - 1 - 2s \cdot Y(s) + 2Y(s) = \dfrac{1}{s+1}$ $\Rightarrow\ Y(s) \cdot [s^2 - 2s + 2] - 1 = \dfrac{1}{s+1}$ $\Rightarrow\ Y(s) = \dfrac{1}{(s+1)(s^2-2s+2)} + \dfrac{1}{s^2-2s+2}$	Laplace transform both sides Rearrange

4	Let, $\dfrac{1}{(s+1)(s^2-2s+2)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2-2s+2}$ $\Rightarrow 1 = A(s^2-2s+2) + (Bs+C)(s+1)$ $\Rightarrow 1 = As^2 - 2As + 2A + Bs^2 + Bs + Cs + C$ $\Rightarrow 1 = (A+B)s^2 + (B+C-2A)s + 2A + C$ $\Rightarrow A + B = 0$ $B + C - 2A = 0$ $2A + C = 1$ $\Rightarrow A = -B$ $\Rightarrow B + C - 2(-B) = 0$ $2(-B) + C = 1$ $\Rightarrow B + C + 2B = 0$ $-2B + C = 1$ $\Rightarrow 5B = -1$ $\Rightarrow B = -\dfrac{1}{5},\ A = \dfrac{1}{5},\ C = \dfrac{3}{5}$ Hence, $\dfrac{1}{(s+1)(s^2-2s+2)} = (\dfrac{1}{5}) \cdot \dfrac{1}{s+1} + (\dfrac{1}{5}) \cdot \dfrac{-s+3}{s^2-2s+2}$	Separate by partial fractions Multiply both sides by $(s+1)(s^2-2s+2)$ Compare coefficients Solve simultaneous equations

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$Y(s)=(\frac{1}{5})\cdot\frac{1}{s+1}+(\frac{1}{5})\cdot\frac{-s+3}{s^2-2s+2}+\frac{1}{s^2-2s+2}$

$\Rightarrow Y(s)=(\frac{1}{5})\cdot\frac{1}{s+1}+(\frac{1}{5})\cdot\frac{-s+3}{(s-1)^2+1}+\frac{1}{(s-1)^2+1}$

$\Rightarrow Y(s)=(\frac{1}{5})\cdot[\frac{1}{s+1}-\frac{s-3}{(s-1)^2+1}+5\cdot\frac{1}{(s-1)^2+1}]$

$\Rightarrow Y(s)=(\frac{1}{5})\cdot[\frac{1}{s+1}-(\frac{s-1}{(s-1)^2+1}-\frac{2}{(s-1)^2+1})+$
$5\cdot\frac{1}{(s-1)^2+1}]$

$\Rightarrow Y(s)=(\frac{1}{5})\cdot[\frac{1}{s+1}-\frac{s-1}{(s-1)^2+1}+7\cdot\frac{1}{(s-1)^2+1}]$

$\Rightarrow y=(\frac{1}{5})\cdot[e^{-t}-e^t cos(t)+7e^t sin(t)]$

Rearrange
Inverse Laplace

6.1 Page 324 Q38

Since $g(t)=\int_0^t f(\tau)\,d\tau$ we have that $g'(t)=f(t)$ for all $t\geq 0$.

Hence $\mathcal{L}[f(t)]=\mathcal{L}[g'(t)]=s\mathcal{L}[g(t)]-g(0)$.

Since $g(0)=\int_0^0 f(\tau)\,d\tau=0$, $\mathcal{L}[f(t)]=F(s)$ and $\mathcal{L}[g(t)]=G(s)$ we have that $F(s)=sG(s)$ and then $\frac{F(s)}{s}=G(s)$.